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SOME COUPLED FIXED POINT RESULTS ON N_p-CONE METRIC SPACES OVER BANACH ALGEBRA

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Abstract: The main aim of this paper is to introduce the notion of N_p-cone metric spaces over Banach algebra. We define T-contraction map and study some coupled common fixed point theorems in the new setup. Finally, we state an example as an application to illustrate our obtained results. Our results generalize and extend some well-known results in the literature.

Keywords and Phrases: N_p-cone metric spaces over Banach algebra, c-sequence, generalized Lipschitz map, coupled fixed points.

2020 Mathematics Subject Classification: 46B20, 46B40, 46J10, 54A05, 47H10.

1. Introduction

Matthews [23], in 1994, introduced the concept of partial metric spaces, which differ from the usual metric spaces by the notable property that the self-distance of a point need not be zero.

In recent years, one of the most active research areas in fixed point theory has been the study of fixed points in cone metric spaces without assuming normality. The notion of cone metric spaces was first introduced by Huang and Zhang [17] as a generalization of metric spaces, where the set of real numbers is replaced by a Banach space E. They extended the Banach contraction principle to cone metric spaces over a normal solid cone. Later, in 2008, Rezapour and Hamlbarani [25] relaxed the normality condition of the cone and obtained more general results, thereby extending the findings of [17].

In another direction, Malviya et al. [22] introduced the concept of N-cone metric spaces as a generalization of D^*-metric spaces [1]. They further defined asymptotically regular sequences and map, proving several fixed point theorems for self-map. Continuing along this line of investigation, Fernandez et al. [12] established the Banach contraction principle and other fixed point theorems in N-cone metric spaces for contractive map.

More recently, some authors (see, e.g., [4]) have reported that fixed point results in cone metric spaces are, in fact, equivalent to the corresponding results in usual metric spaces. Motivated by these discussions, Xu and Radenovic [28] extended the work of Liu and Xu [21] on cone metric spaces over Banach algebras by eliminating the assumption of cone normality. Building upon this framework, Fernandez et al. [6] introduced a generalization of N-cone metric spaces over Banach algebras and established fixed point results for generalized Lipschitz map. Further motivation for the present study is drawn from related investigations reported in [5, 7, 8, 10, 11, 13, 14].

The notion of coupled fixed points was introduced by Guo and Lakshmikantham [15] in 1987. Subsequently, Bhaskar and Lakshmikantham [3] introduced the mixed monotone property for contractive operators of the type $F : B \times B \to B$, where B is a partially ordered metric space. Later, Shatanawi [27] proved coupled coincidence fixed point theorems in cone metric spaces without requiring cone normality.

This paper is organized into six sections. Following the introduction, Section 2 presents the preliminaries, including essential definitions and results required in the new framework. In Section 3, we introduce the concept of N_p-cone metric spaces over Banach algebras, which generalizes both N-cone metric spaces over

Banach algebras and p-metric spaces. Section 4 is devoted to a discussion of their topological properties. In Section 5, we define generalized Lipschitz map, while Section 6 establishes several coupled common fixed point theorems. Finally, we illustrate our results with a concrete example. The theorems presented here extend and generalize the results of [29].

2. Preliminaries

We recall a few essential definitions that are used throughout this paper.

Let A be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A,\ \beta \in \mathbb{R}$)

1. $(xy)z = x(yz)$,
2. $x(y+z) = xy + xz$ and $(x+y)z = xy + xz$,
3. $\beta(xy) = (\beta x)y = x(\beta y)$,
4. $\|xy\| \leq \|x\| \, \|y\|$.

In the sequel, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer the reader to [26].

Proposition 2.1. [26] *Let A be a Banach algebra with a unit e, and $x \in A$. If the spectral radius $r(x)$ of x is less than 1, i.e.,*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. In fact,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Remark 2.2. *From* [26]*, the spectral radius $r(x)$ of x satisfies $r(x) \leq \|x\|$ for all $x \in A$, where A is a Banach algebra with a unit e.*

Remark 2.3. [28] *In Proposition 2.1, if we replace $r(x) < 1$ by $\|x\| \leq 1$, then we obtain the same result.*

Remark 2.4. [28] *If $r(x) < 1$ then $\|x^n\| \to 0 (n \to \infty)$.*

A subset P of A is called a cone of A if

1. *P is non-empty closed and $\{\theta, e\} \subset P$;*
2. *$\gamma P + \psi P \subset P$ for all non-negative real numbers γ, ψ;*
3. *$P^2 = PP \subset P$;*
4. *$P \cap (-P) = \{\theta\}$,*

where θ denotes the null of the Banach algebra A. For a given cone $P \subset A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. If $intP \neq \varnothing$ then P is called a solid cone.

The cone P is called normal if there is a number $N > 0$ such that, for all $x, y \in A$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq N \|y\| .$$

The least positive number satisfying above is called the normal constant of P [17].

In the following assume that P is a solid cone in A and $\preceq$ is the partial ordering with respect to P.

Definition 2.5. ([17, 21]) *Let X be a nonempty set. Suppose a map $d : X \times X \to A$ satisfies*

1. *$\theta \prec d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
3. *$d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra A.

For other definitions and related results on cone metric space over Banach algebra we refer to [21].

Definition 2.6. ([23]) *A partial metric on a nonempty set X is a function $\xi_p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$, the following conditions hold:*

1. *$x = y \Leftrightarrow \xi_p(x, x) = \xi_p(x, y) = \xi_p(y, y)$,*
2. *$\xi_p(x, x) \leq \xi_p(x, y)$,*
3. *$\xi_p(x, y) = \xi_p(y, x)$,*
4. *$\xi_p(x, y) \leq \xi_p(x, z) + \xi_p(z, y) - \xi_p(z, z)$.*

The pair (X, ξ_p) is called a partial metric space. It is clear that, if $\xi_p(x, y) = 0$, then from (1) and (2) $x = y$. But if $x = y, \xi_p(x, y)$ may not be 0.

Definition 2.7. ([22]) *Let X be a nonempty set and E be a real Banach space. A map $N : X^3 \to E$ is called an N-cone metric on X if for any $x, y, z, a \in X$, the following conditions are satisfied:*

1. $\theta \leq N(x, y, z)$,
2. $N(x, y, z) = \theta$ *if and only if* $x = y = z$,
3. $N(x, y, z) \leq N(x, x, a) + N(y, y, a) + N(z, z, a)$.

Then the pair (X, N) is called an N-cone metric space.

For more definitions and subsequent results on N-cone metric spaces see [22].

Definition 2.8. ([6]) *An N-cone metric on a nonempty set X is a map $N : X^3 \to A$ such that for all $x, y, z, a \in X$, the following conditions hold:*

1. $\theta \preceq N(x, y, z)$,
2. $N(x, y, z) = \theta$ *if and only if* $x = y = z$,
3. $N(x, y, z) \preceq N(x, x, a) + N(y, y, a) + N(z, z, a)$.

Then the pair (X, N) is called an N-cone metric space over Banach algebra A.

3. N_p-cone metric space over Banach algebra

In this section, we present the concept of N_p-cone metric space over Banach algebra and study some of its properties needed later.

Definition 3.1. *An N_p-cone metric on a nonempty set X is a map $N_p : X^3 \to A$ such that for all $x, y, z, a \in X$,*
(N_{p1}) $x = y = z \Leftrightarrow N_p(x, x, x) = N_p(y, y, y) = N_p(z, z, z) = N_p(x, y, z)$,
(N_{p2}) $\theta \preceq N_p(x, x, x) \preceq N_p(x, x, y) \preceq N_p(x, y, z)$, *for all* $x, y, z \in X$ *with* $x \neq y \neq z$;
(N_{p3}) $N_p(x, x, y) = N_p(y, y, x)$,
(N_{p4}) $N_p(x, y, z) \preceq N_p(x, x, a) + N_p(y, y, a) + N_p(z, z, a) - N_p(a, a, a)$.
The pair (X, N_p) is called an N_p-cone metric space over Banach algebra A.

Remark 3.2. *It is clear that in an N_p-cone metric space over Banach algebra, if $x, y, z \in X$ and $N_p(x, y, z) = \theta$, then $x = y = z$, but the converse may not be true.*

We now present some examples of N_p-cone metric space over Banach algebra, which is neither an N-cone metric space over Banach algebra [6] nor an N_b-cone metric space over Banach algebra [9].

Example 3.3. Let $A = C^1_R[0,1]$ and define a norm on A by $\|x\| = \|x\|_\infty + \|x'\|_\infty$, for $x \in A$. Define multiplication in A as just point wise multiplication. Then A is a real unit Banach algebra with unit $e = 1$. Set $P = \{x \in A : x \geq 0\}$ is a cone in A. Moreover, P is not normal (see [25]). Let $X = [0, \infty)$. Define a map $N_p : X^3 \to A$ by $N_p(x, y, z)(t) = (\max\{x, z\} + \max\{y, z\})e^t$ for all $x, y, z \in X$. Then (X, N_p) is an N_p-cone metric space over Banach algebra. But it is not an N-cone metric space over Banach algebra since for any $t > 0$ we have $N_p(x, x, x)(t) = 2xe^t \neq \theta$.

Definition 3.4. *Let (X, N_p) be an N_p-cone metric space over Banach algebra A. Then for $x \in X$ and $c > \theta$,*

$$B_{N_p}(x, c) = \Big\{y \in X : N_p(x, x, y) \ll N_p(x, x, x) + c\Big\}.$$

4. Topology on N_p-cone metric space over Banach algebra

In this section, we discuss the topology on N_p-cone metric space over Banach algebra A.

Definition 4.1. *Let (X, N_p) be an N_p-cone metric space over Banach algebra A. For each $x \in X$ and each $\theta \ll c$, put $B_{N_p}(x, c) = \Big\{y \in X : N_p(x, x, y) \ll N_p(x, x, x) + c\Big\}$ and put $\mathfrak{B} = \{B_{N_p}(x, c) : x \in X \text{ and } \theta \ll c\}$. Then $\mathfrak{B}$ is a subbase for some topology τ on X.*

Remark 4.2. *Let (X, N_p) be an N_p-cone metric space over Banach algebra A. In this paper, τ denotes the topology on X, $\mathfrak{B}$ denotes a subbase for the topology on τ and $\mathfrak{B}_{N_p}(x, c)$ denotes the N_p-ball in (X, N_p), which are described in Definition 4.1. Moreover U denotes the base generated by the subbase $\mathfrak{B}$.*

Theorem 4.3. *Let (X, N_p) be an N_p-cone metric space over Banach algebra A and P be a solid cone in A. Then, (X, N_p) is a Hausdorff space.*

Proof. Let (X, N_p) be an N_p-cone metric space over Banach algebra and let $x, y \in X$ with $x \neq y$. Let $N_p(x, x, y) = c$. Suppose $U_1 = B\Big(x, \frac{c}{4}\Big)$ and $U_2 = B\Big(y, \frac{c}{2}\Big)$. Then $x \in U_1$ and $y \in U_2$. We claim that $U_1 \cap U_2 = \phi$. If not there exists $z \in U_1 \cap U_2$. But then

$$N_p(x, x, z) \prec \frac{c}{4} \text{ and } N_p(y, y, z) \prec \frac{c}{2}.$$

So, we get

$$\begin{aligned} c = N_p(x, x, y) &\preceq \Big[N_p(x, x, z) + N_p(x, x, z) + N_p(y, y, z)\Big] - N_p(z, z, z) \\ &\prec 2N_p(x, x, z) + N_p(y, y, z) \end{aligned}$$

$$\begin{aligned} &\preceq 2\frac{c}{4} + \frac{c}{2} \\ &\prec c. \end{aligned}$$

That is, $c < c$, which is a contradiction. Hence $U_1 \cap U_2 = \phi$ and E is a Hausdorff space.

Now, we define θ-Cauchy sequence and convergent sequence in N_p-cone metric space over Banach algebra A.

Definition 4.4. *Let (X, N_p) be an N_p-cone metric space over Banach algebra A. A sequence $\{x_n\}$ in (X, N_p) converges to a point $x \in X$ whenever for every $c \gg \theta$ there is a natural number N such that $N_p(x_n, x, x) \ll c$ for all $n \geq N$. We denote this by*

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \ (n \to \infty).$$

Definition 4.5. *Let (X, N_p) be an N_p-cone metric space over Banach algebra A. A sequence $\{x_n\}$ in X is said to be a θ-Cauchy sequence in (X, N_p), i.e., if for every $c \gg \theta$ there exists $n_0 \in N$ such that, $N_p(x_n, x_m, x_m) \ll c$ for all $n, m \geq n_0$.*

Definition 4.6. *Let (X, N_p) be an N_p-cone metric space over Banach algebra A. Then X is said to be θ-complete if every θ-Cauchy sequence $\{x_n\}$ in (X, N_p) converges to $x \in X$ such that $N_p(x, x, x) = \theta$.*

Definition 4.7. *Let (X, N_p) and (X', N_p') be N_p-cone metric spaces over Banach algebra A. Then a map $g : X \to X'$ is said to be continuous at a point $x \in X$ if it is sequentially continuous at x, that is, whenever $\{x_n\}$ is convergent to x we have $\{gx_n\}$ is convergent to $g(x)$.*

5. T-Contraction map

In this section, we introduce the concept of T-contraction map on N_p-cone metric space over Banach algebra with an example.

Definition 5.1. *Let (X, N_p) be an N_p-cone metric space over Banach algebra A and $T : X \to X$ be a map. A map $F : X \times X \to X$ is said to be a T-contraction if there exists a vector $q \in P$ with $r(q) < 1$ such that for all $x, y \in X$, we have*

$$N_p(TF(x,x), TF(x,x), TF(y,y)) \preceq qN_p(Tx, Tx, Ty).$$

Remark 5.2. *A contraction is T-contraction since it suffices to take $T = I$, where I is the identity map on X.*

Example 5.3. *Let $X = R$ and $A = R$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$. Define multiplication in usual way. Let $P = \{x \in A : x \geq 0\}$ be a cone in A. Moreover, P*

is not normal (see [25]) and A *is a real unital Banach algebra with unit* $e = 1$*. Let* $X = [0, \infty)$*. Define* $N_p : X^3 \to A$ *by* $N_p(x, y, z)(t) = \Big(\max\{x, z\} + \max\{y, z\}\Big)e^t$ *for all* $x, y, z \in X$*. Then* (X, N_p) *is an* N_p*-cone metric space over Banach algebra. Now define the maps* $F : X \times X \to X$ *and* $g : X \to X$ *by* $F(x, y) = \frac{x+y}{8}$*, for all* $x, y \in X$ *and* $g(x) = \frac{x}{2}$ *for all* $x \in X$*. Then we have,*

$$\begin{aligned}
N_p(TF(x,x),&TF(x,x), TF(y,y))(t)\\
&= \Bigg(\max\{TF(x,x), TF(y,y)\} + \max\{TF(x,x), TF(y,y)\}\Bigg)e^t\\
&= 2\Bigg(\max\{TF(x,x), TF(y,y)\}\Bigg)e^t\\
&= 2\Bigg(\max\{\frac{2x}{16}, \frac{2y}{16}\}\Bigg)e^t\\
&= 2\Bigg(\max\{\frac{x}{8}, \frac{y}{8}\}\Bigg)e^t\\
&\preceq \frac{1}{4}[2\max\{\frac{x}{2}, \frac{y}{2}\}]e^t\\
&< \frac{1}{4}N_p(Tx, Tx, Ty)(t)
\end{aligned}$$

where $q = \frac{1}{4}$*. Clearly,* F *is a* T*-contraction in* X*.*

Definition 5.4. ([3]) *An element* $(x, y) \in X \times X$ *is called a coupled fixed point of a map* $F : X \times X \to X$ *if* $F(x, y) = x$ *and* $F(y, x) = y$*.*
Note that if (x, y) is a coupled fixed point of F, then (y, x) is also a coupled fixed point of F.

Definition 5.5. ([20]) *An element* $(x, y) \in X \times X$ *is called a coupled coincidence point of maps* $F : X \times X \to X$ *and* $h : X \to X$ *if* $F(x, y) = h(x) = x$ *and* $F(y, x) = h(y) = y$*.*

Definition 5.6. ([2]) *Let* h_1, h_2 *be self-maps on* X*. Then* $x \in X$ *is called a coincidence point of the pair* (h_1, h_2) *if* $h_1x = h_2x$*, and* $z \in X$ *is called a point of the coincidence of pair* (h_1, h_2) *if* $h_1x = h_2x = z$*.*
Now we review some facts on c-sequence theory.

Definition 5.7. ([19]) *Let* P *be a solid cone in a Banach space* E*. A sequence* $\{x_n\} \subset P$ *is said to be a c-sequence if for each* $c \gg \theta$ *there exists a natural number*

N such that $x_n \ll c$ for all $n > N$.

Lemma 5.8. ([28]) *Let P be a solid cone over a Banach algebra A. Suppose that $z \in P$ and $\{x_n\}$ is a c-sequence in P. Then $\{zx_n\}$ is a c-sequence.*

Lemma 5.9. ([26]) *Let A be a unital Banach algebra with a unit $e, x \in A$, then $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and the spectral radius $r(x)$ satisfies*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}}.$$

If $r(x) < |\beta|$, then $(\beta e - x)$ is invertible in A, moreover,

$$(\beta e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\beta^{i+1}}$$

where β is a constant.

Lemma 5.10. ([26]) *Let A be a unital Banach algebra with a unit $e, x_1, x_2 \in A$. If x_1 commutes with x_2, then*

$$r(x_1 + x_2) \leq r(x_1) + r(x_2), \;\; r(x_1x_2) \leq r(x_1)r(x_2).$$

Lemma 5.11. ([16]) *Let A be a unital Banach algebra with a unit e and $x \in A$. If β is a constant and $r(x) < |\beta|$, then*

$$r\Big((\beta e - x)^{-1}\Big) \leq \frac{1}{|\beta| - r(x)}.$$

Lemma 5.12. ([16]) *Let A be a unital Banach algebra with a unit e and P be a solid cone in A. Let $a, x, l \in P$ satisfy $l \preceq x$ and $a \preceq la$. If $r(x) < 1$, then $a = \theta$.*

Lemma 5.13. ([16]) *If E is a real Banach space with a solid cone P and $\{x_n\} \subset P$ is a sequence with $\|x_n\| \to 0$ $(n \to \infty)$, then $\{x_n\}$ is a c-sequence.*

6. Application in fixed point theory

In this section, as an application we prove fixed point results for T-contraction map with an example to illustrate our main results.

Theorem 6.1. *Let (X, N_p) be a θ-complete N_p-cone metric space over Banach algebra A and P be a solid cone in A. Let $\alpha, \beta \in P$ with $\rho[\alpha + \beta] < 1$. Suppose that $F : X \times X \to X$ is a T-contraction where $T : X \to X$ is a bijective map satisfying*

$$N_p(TF(x,y), TF(x,y), TF(u,v)) \preceq \alpha N_p(Tx,Tx,Tu) + \beta N_p(Ty,Ty,Tv) \quad (6.1)$$

for all $x, y, u, v \in X$. Then there exist unique x^*, y^* such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and we denote

$$x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0),\ y_{n+1} = F(y_n, x_n) = F^{n+1}(y_0, x_0),$$

for all $n \in N$. Now according to (6.1), we have

$$\begin{aligned} N_p(Tx_n, Tx_n, Tx_{n+1}) &= N_p(TF(x_{n-1}, y_{n-1}), TF(x_{n-1}, y_{n-1}), TF(x_n, y_n)) \\ &\preceq \alpha N_p(Tx_{n-1}, Tx_{n-1}, Tx_n) + \beta N_p(Ty_{n-1}, Ty_{n-1}, Ty_n) \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} N_p(Ty_n, Ty_n, Ty_{n+1}) &= N_p(TF(y_{n-1}, x_{n-1}), TF(y_{n-1}, x_{n-1}), TF(y_n, x_n)) \\ &\preceq \alpha N_p(Ty_{n-1}, Ty_{n-1}, Ty_n) + \beta N_p(Tx_{n-1}, Tx_{n-1}, Tx_n). \end{aligned} \tag{6.3}$$

Let $d_n = N_p(Tx_n, Tx_n, Tx_{n+1}) + N_p(Ty_n, Ty_n, Ty_{n+1})$. Then, adding up (6.2) and (6.3), we obtain

$$\begin{aligned} d_n &\preceq (\alpha + \beta)[N_p(Tx_{n-1}, Tx_{n-1}, Tx_n) + N_p(Ty_{n-1}, Ty_{n-1}, Ty_n)] \\ &\preceq (\alpha + \beta) d_{n-1} \end{aligned}$$

which implies that

$$d_n \preceq h d_{n-1}$$

where $h = \alpha + \beta$, with $\rho(h) < 1$. It is evident that

$$\theta \preceq d_n \preceq h\, d_{n-1} \preceq \cdots \preceq h^n\, d_0. \tag{6.4}$$

Without loss of generality, we assume that $d_0 > 0$. Otherwise (x_0, y_0) is a coupled fixed point of F. If $m > n$, then we have

$$\begin{aligned} N_p(Tx_n, Tx_n, Tx_m) &\preceq N_p(Tx_n, Tx_n, Tx_{n+1}) + N_p(Tx_n, Tx_n, Tx_{n+1}) \\ &\quad + N_p(Tx_{n+1}, Tx_{n+1}, Tx_m) - N_p(Tx_{n+1}, Tx_{n+1}, Tx_{n+1}) \\ &\preceq 2N_p(Tx_n, Tx_n, Tx_{n+1}) + N_p(Tx_{n+1}, Tx_{n+1}, Tx_m) \\ &\preceq 2N_p(Tx_n, Tx_n, Tx_{n+1}) + N_p(Tx_{n+1}, Tx_{n+1}, Tx_{n+2}) \\ &\quad + N_p(Tx_{n+1}, Tx_{n+1}, Tx_{n+2}) + N_p(Tx_{n+2}, Tx_{n+2}, Tx_m) \\ &\quad - N_p(Tx_{n+2}, Tx_{n+2}, Tx_{n+2}) \\ &\preceq 2N_p(Tx_n, Tx_n, Tx_{n+1}) + 2N_p(Tx_{n+1}, Tx_{n+1}, Tx_{n+2}) \end{aligned}$$

$$
\begin{aligned}
&+ N_p(Tx_{n+2}, Tx_{n+2}, Tx_m)\\
&\vdots\\
&\preceq 2N_p(Tx_n, Tx_n, Tx_{n+1}) + 2N_p(Tx_{n+1}, Tx_{n+1}, Tx_{n+2}) + \cdots\\
&+ 2N_p(Tx_{m-2}, Tx_{m-2}, Tx_{m-1}) + N_p(Tx_{m-1}, Tx_{m-1}, Tx_m)\\
&\preceq 2[N_p(Tx_n, Tx_n, Tx_{n+1}) + N_p(Tx_{n+1}, Tx_{n+1}, Tx_{n+2}) + \cdots\\
&+ N_p(Tx_{m-2}, Tx_{m-2}, Tx_{m-1}) + N_p(Tx_{m-1}, Tx_{m-1}, Tx_m)] \qquad (6.5)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
N_p(Ty_n, Ty_n, Ty_m) \preceq{}& 2[N_p(Ty_n, Ty_n, Ty_{n+1}) + N_p(Ty_{n+1}, Ty_{n+1}, Ty_{n+2}) + \cdots\\
&+ N_p(Ty_{m-2}, Ty_{m-2}, Ty_{m-1}) + N_p(Ty_{m-1}, Ty_{m-1}, Ty_m)]. \qquad (6.6)
\end{aligned}
$$

Adding up (6.5) and (6.6) and using (6.4), we have

$$
\begin{aligned}
N_p(Tx_n, Tx_n, Tx_m) + N_p(Ty_n, Ty_n, Ty_m) &\preceq 2d_n + 2d_{n+1} + ... + 2d_{m-1}\\
&\preceq 2[h^n + h^{n+1} + ... + h^{m-1}]d_0\\
&\preceq 2h^n[e + h + ... + h^{m-n-1}]d_0\\
&< 2h^n(e-h)^{-1}d_0.
\end{aligned}
$$

Since $\rho(h) = \rho(\alpha + \beta) < 1$, by Remark 2.4, we get $\|h^n\| \to 0$, and with Lemma 5.4, we have $2h^n d_0$ is a c-sequence. Next by using Lemma 5.8 and Lemma 5.10 we conclude that

$$\lim_{n\to\infty} N_p(Tx_n, Tx_n, Tx_m) = \lim_{n,m\to\infty} N_p(Ty_n, Ty_n, Ty_m) = \theta.$$

By the similar arguments as above, we can show that

$$\lim_{n\to\infty} N_p(Tx_m, Tx_m, Tx_n) = \lim_{n,m\to\infty} N_p(Ty_m, Ty_m, Ty_n) = \theta.$$

Thus, Tx_n and Tx_m are θ-Cauchy sequences in X. The completeness of X gives that there exist $x^*, y^* \in X$ such that

$$
\begin{aligned}
&\lim_{n\to\infty} N_p(TF^n(x_0, y_0), TF^n(x_0, y_0), x^*)\\
&\qquad = \lim_{n,m\to\infty} N_p(TF^n(x_0, y_0), TF^n(x_0, y_0), TF^m(x_0, y_0))\\
&\qquad = N_p(x^*, x^*, x^*) = \theta
\end{aligned}
$$

and

$$\begin{aligned}\lim_{n\to\infty} N_p(TF^n(y_0,x_0),&TF^n(y_0,x_0),y^*)\\ &= \lim_{n,m\to\infty} N_p(TF^n(y_0,x_0),TF^n(y_0,x_0),TF^m(y_0,x_0))\\ &= N_p(y^*,y^*,y^*)=\theta.\end{aligned}$$

We shall prove $F(x^*,y^*)=x^*$. Consider,

$$\begin{aligned}N_p(TF(x^*,y^*),TF(x^*,y^*),Tx^*) &\preceq N_p(TF(x^*,y^*),TF(x^*,y^*),TF(x_n,y_n))\\ &\quad + N_p(TF(x^*,y^*),TF(x^*,y^*),TF(x_n,y_n))\\ &\quad + N_p(Tx^*,Tx^*,TF(x_n,y_n))\\ &\quad - N_p(TF(x_n,y_n),TF(x_n,y_n),TF(x_n,y_n))\\ &\preceq 2N_p(TF(x^*,y^*),TF(x^*,y^*),TF(x_n,y_n))\\ &\quad + N_p(Tx^*,Tx^*,TF(x_n,y_n))\\ &\preceq 2[\alpha N_p(Tx^*,Tx^*,Tx_n)+\beta N_p(Ty^*,Ty^*,Ty_n)]\\ &\quad + N_p(Tx_{n+1},Tx_{n+1},Tx^*).\end{aligned}$$

From the surjective property of T and Lemma 5.8, it follows that $N_p(TF(x^*,y^*), TF(x^*,y^*),Tx^*)=\theta$, that is, $TF(x^*,y^*)=Tx^*$. Since T is one-to-one, $F(x^*,y^*)=x^*$. Similarly, we can get $F(y^*,x^*)=y^*$. Therefore, (x^*,y^*) is a coupled fixed point of F. Now, if (x',y') is another coupled fixed point of F, then

$$\begin{aligned}N_p(Tx^*,Tx^*,Tx') &= N_p(TF(x^*,y^*),TF(x^*,y^*),TF(x',y'))\\ &\preceq \alpha N_p(Tx^*,Tx^*,Tx')+\beta N_p(Ty^*,Ty^*,Ty') \qquad (6.7)\end{aligned}$$

and

$$\begin{aligned}N_p(Ty^*,Ty^*,Ty') &= N_p(TF(y^*,x^*),TF(y^*,x^*),TF(y',x'))\\ &\preceq \alpha N_p(Ty^*,Ty^*,Ty')+\beta N_p(Tx^*,Tx^*,Tx'). \qquad (6.8)\end{aligned}$$

Adding up (6.7) and (6.8), we have

$$\begin{aligned}N_p(Tx^*,Tx^*,Tx')&+N_p(Ty^*,Ty^*,Ty')\\ &\preceq (\alpha+\beta)[N_p(Tx^*,Tx^*,Tx')+N_p(Ty^*,Ty^*,Ty')]\\ &\preceq h[N_p(Tx^*,Tx^*,Tx')+N_p(Ty^*,Ty^*,Ty')]. \qquad (6.9)\end{aligned}$$

Since $\rho(h)=\rho(\alpha+\beta)<1$, it follows from (6.9) that $N_p(Tx^*,Tx^*,Tx')+N_p(Ty^*, Ty^*,Ty')=\theta$. Hence $N_p(Tx^*,Tx^*,Tx')=N_p(Ty^*,Ty^*,Ty')=\theta$. That is, $Tx^*=$

$Tx^{'}$ and $Ty^{*} = Ty^{'}$. As T is one to one, we have $(x^{*}, y^{*}) = (x^{'}, y^{'})$. Thus F has a coupled fixed point.

Corollary 6.2. *Suppose that (X, N_p) is a θ-complete N_p-cone metric space over Banach algebra A and P is a solid cone in A, and $T : X \to X$ is a one to one and surjective map. Then each T-contraction on X has a unique fixed point.*

The following example is an illustration of Theorem 6.1.

Example 6.3. Let $X = C[-1, 1]$ and A be the set of all real valued function on the interval $[0, 1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and multiplication in the usual way. Let $P = \Big\{x \in A : x(t) \geq 0, t \in [0, 1]\Big\}$. It is clear that P is a non normal cone and A is a Banach algebra with a unit $e = 1$. Define a map $N_p : X^3 \to A$ by

$$N_p(x, y, z)(t) = \Big(\max\{x, z\} + \max\{y, z\}\Big)e^t$$

for all $x, y, z \in X$. Then (X, N_p) is a complete N_p-cone metric space over Banach algebra A. Now define the maps $F : X \times X \to X$ by $F(x, y) = \frac{x+y}{6}$ for all $x, y \in X$ and also define $T : X \to X$ by $T(x) = \frac{x}{3}$ for all $x \in X$.
Also,

$$\begin{aligned} N_p(TF(x, y), TF(x, y), TF(u, v))(t) &= \Big(\max\{\frac{x+y}{18}, \frac{u+v}{18}\} + \max\{\frac{x+y}{18}, \frac{u+v}{18}\}\Big)e^t \\ &= 2\max\{\frac{x+y}{18}, \frac{u+v}{18}\}e^t \\ &= \frac{2}{18}\max\{x+y, u+v\}e^t \\ &\preceq \frac{1}{18}[\max\{x, u\} + \max\{y, v\}]e^t \end{aligned}$$

where $\alpha = \beta = \frac{1}{18}$, then the condition of Corollary 6.2 holds trivially and $(0, 0)$ is a unique coupled fixed point of F.

7. Conclusion

In this paper, we introduced the notion of N_p-cone metric space over Banach algebras. We obtained fixed point results for several contractions that generalize previous results. Also, we presented an example to illustrate one of the main result. As perspectives for this work, we can think to prove Theorem 6.1 without using one of the conditions namely injectivity or surjectivity.

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